Mathematics and the Mind

LSE 10-10-03

1. Isn't it a bad thing to be deceived about the truth, and a good thing to know what the truth is? — Plato.

2 So. Truth is valuable (truth per se or interesting truth?)

But certifiable truth is hard to come by. (cp theology, natural & social sciences)

3. The case of mathematics:
Does truth equate with proof?

NO Gödel & Tarski have shown how truth outruns proof even in the case of the arithmetic of the natural numbers 0, 1, 2, ...

4. The argument that ascertainable truth outruns proof in mathematics has been used by Lucas and Penrose to argue that human minds can do ~~more than~~ things which computers can't do, and hence that minds can't be (digital) machines.

5. I want to look at this argument and tease out exactly what the claim amounts to using much simpler examples than those employed by Lucas & Penrose

* Students effectively

So addition is reduced to adding one and one -- a suitable number of times, and multiplication to repeated addition.

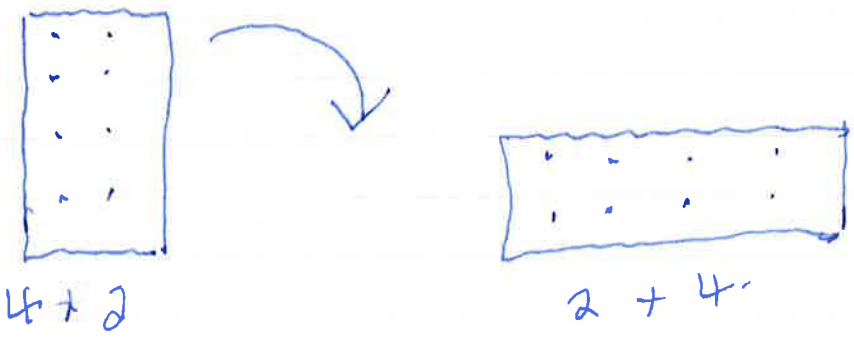
We now have the machinery to check out some unchecked facts like $346 = 643$ or $279 + 681 = 681 + 279$.

6. So I am asking the following question.

What sort of truths in arithmetic sustain the Lucas-Penrose argument, and how do we know them to be true, if we can't prove them!

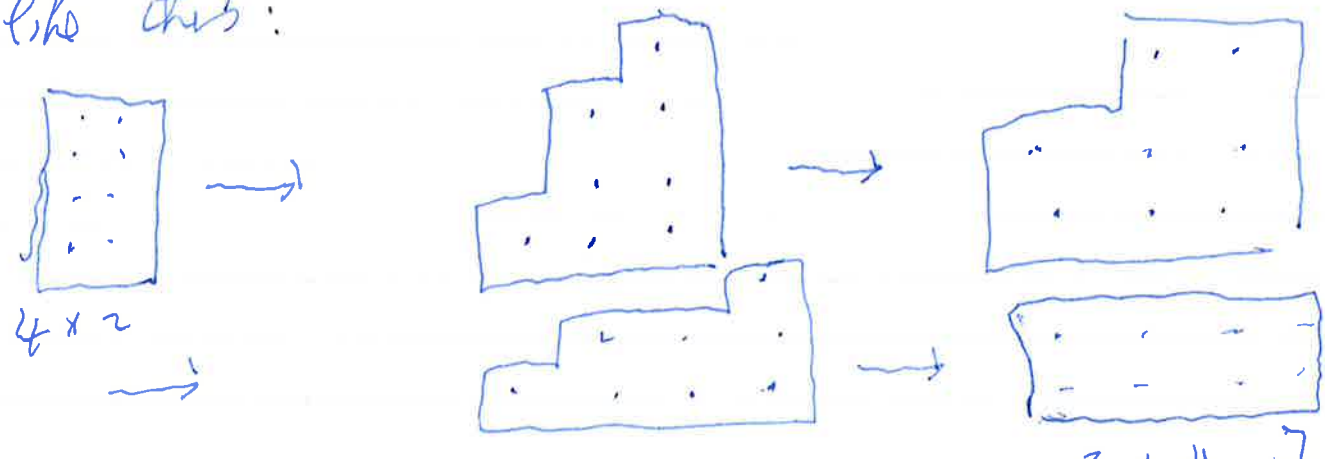
7. Soules arithmetic This is the sort of arithmetic you learnt at school. The basic operation is moving from one number to its next and repeating to perform a finite number of times. addition and multiplication are defined recursively in the usual way. *Twistor

[Ex In Soules arithmetic we can prove statements like ~~2+4 = 4+2~~. 4+2 = 2+4



This has a geometrical flavour.

In purely arithmetical terms proof goes like this:



8. Indeed for any pair of numbers (m, n) we can prove

$$m \times n = n \times m. \quad (\star_2)$$

(1)

This gives a whole array of true statements about natural numbers.

9. But what we can't prove in basic arithmetic is the Commutative Law of Multiplication. In the form

CM For all pairs of numbers (m, n)

$$m \times n = n \times m \quad (\star_3)$$

(2)

Why is this?

Because as the numbers m and n get bigger and bigger the proof increases in length without limit, so there is no finite proof that will work for all pairs of numbers.

10. In other words we cannot make the transition

Any $\rightarrow$ all

which is what is required for (2)
(Or all $\rightarrow$ any which is $\star_3$)

11. Conjecture:

Any plum in this plum pudding is delicious → all the plums in the pudding are delicious.

This is a perfectly valid argument because we are claiming that a typical plum is delicious not in virtue of being that particular plum, but just in virtue of being a plum.

But the corresponding argument

Any pair of numbers has the property ~~that~~ of commuting multiplication → all numbers have the property of commuting multiplication

~~is~~ false because. This is the exact thing as a typical pair of numbers

~~the proof that~~ a particular pair of numbers has the property of commuting multiplication ~~depends~~ depends on which pair of numbers you choose.

12. But although we can't prove (2) we can argue that it is true!

From (1) For any pair of numbers (m, n) } (3)
it is true that $m + n = n + m$.

But (3) is exactly equivalent to (2)

Therefore For all pairs of numbers (m, n) } (4)
$m + n = n + m$

Emelia →

People's response has all the advantages of theft over honest toil (to paraphrase Russell)

13. So in Peano arithmetic the commutative law of multiplication is true but can't be proved.

Notice the proof argument hinges crucially on the fact that the ~~number~~ sequence of numbers is infinite. If there were only finitely many numbers the argument would fail to work.

14. Response. Why not introduce a new axiom to strengthen Peano arithmetic in such a way that the Commutative Law of multiplication (CM) can be proved?

This step was taken by Peano in 1889, and in the resulting system known as Peano arithmetic, CM is both true and provable.

But in 1931 Gödel showed that examples could be given of true but unprovable statements in Peano arithmetic and indeed in any strengthening of Peano arithmetic.

15. But none of these developments bear at all on the question of why we believe CM to be a true property of the natural numbers.

Peano's new axiom is just as much in need of justification as CM itself, (and similar arguments can be made for it).

Lucas's Penrose work with the very careless question of unprovable statements in Peano arithmetic

(N.B. The Gödel proposition can also be used in Gödel arithmetic)

* Incertia

Of course Gödel gave a formal proof that G is unprovable.

I have only given an informal argument that CM is unprovable

My claim is that the axioms of these dynamics can all be framed in the context of finite arithmetic and simple examples like CM.
* Smaller

16 Does an argument show that proofs are not theorems?

For any given unprovable statement we can always work in a stronger system where it becomes provable but no machine can deal with all unprovable statements.

17. Note an argument for the truth of CH makes its semantic conception of truth, and moreover truth of statements which are unprovably quantified over infinite domains.
In such a case do we really have a clear conception of truth?

18 Q: Gentzen's Ramsey centers room for LaLa Land?

19 So the Lucas-Penrose argument is contentious.
In this short post I have pointed to the key issues in terms of which the argument should be assessed.

20. To return to our starting point, we argued that Truth is valuable but ever in mathematics the matter of whether we really know the truth of statements like CH is deeply problematic for the reasons and Rank indicates. ~~constructivism~~ Unless, that is, you are an ~~unconstructivist~~ Platonist in the philosophy of mathematics

Additional comments

1. Presburger's arithmetic (only additions)
   Assumes commutative law as axiom
   cf also Skolem's version using only multiplication
2. Halting problem. Will a Turing Machine halt given an arbitrary input?
3. Oracle computable – generalization of Turing-computable.
4. Grzegorczyk computability.
5. ~~Euthyphro~~ dilemma – Value → Good of
   Good → value